Elastodynamics of Cracked Plates Using Mixed Finite Elements

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Conference in honour of Prof. Dostál, May 25-27: cmse.it4i.cz

Scope of the conference

- Applied mathematics
- Numerical linear algebra
- Optimization methods
- Computational sciences
- High performance computing

Invited plenary speakers

- Rolf Krause (Università della Svizzera italiana)
- Ulrich Langer (Johannes Kepler University Linz)
- Jan Mandel (University of Colorado Denver)
- John Rasmussen (Aalborg University, Denmark)
- François-Xavier Roux (ONERA, University Paris 6)
- Joachim Schöberl (Vienna University of Technology)

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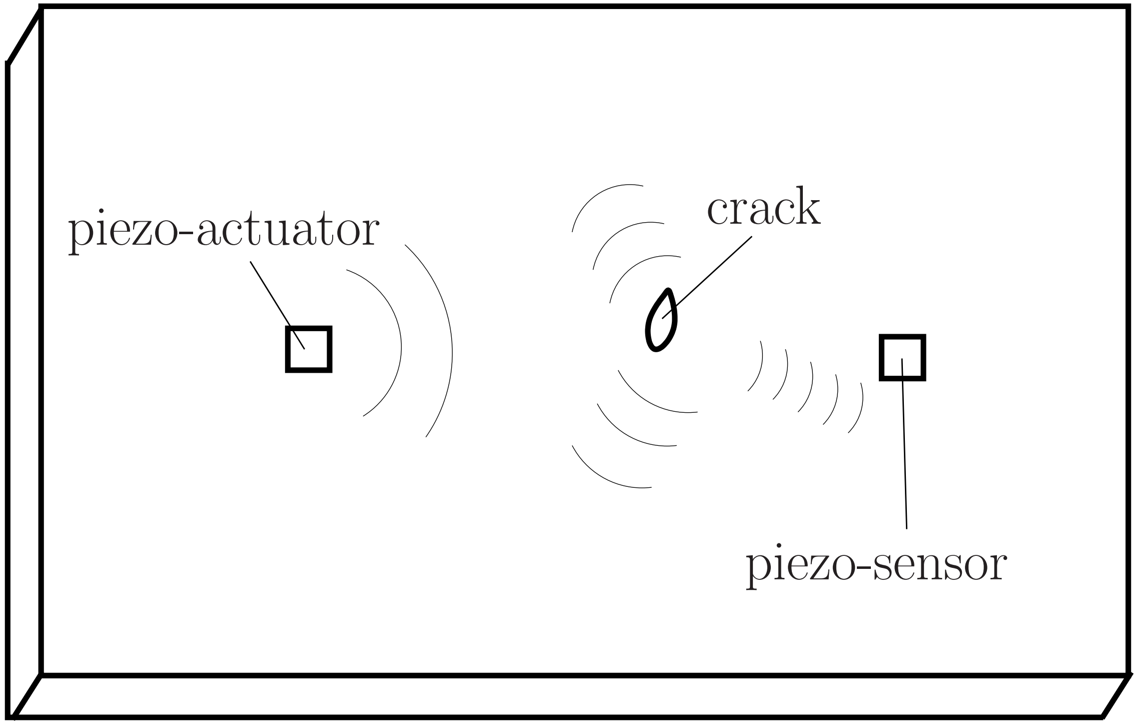
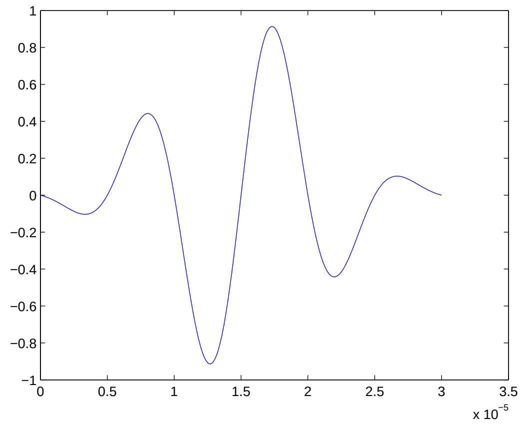
Structural health monitoring of aircrafts





Difficulties: non-harmonic excitation pulses, freq. 10^5 Hz, $T = 10^{-3}$ s, locking effect

Elastodynamics of Cracked Plates
Using Mixed Finite Elements

Outline

- Elastic guided waves in plates
- Displacement finite elements for plates
  - 2d membranes and Kirchhoff's plates
  - 3d bricks enhanced with vertical edge polynomials
- Mixed TD-NNS tensor-product elements
  - Mixed elastic finite elements
  - Tangential displacements normal-normal stresses (TD-NNS)
  - Numerical dispersion analysis
  - Hybridization, parallel preconditioning
  - ...
- Outlook: towards multigrid and DDM

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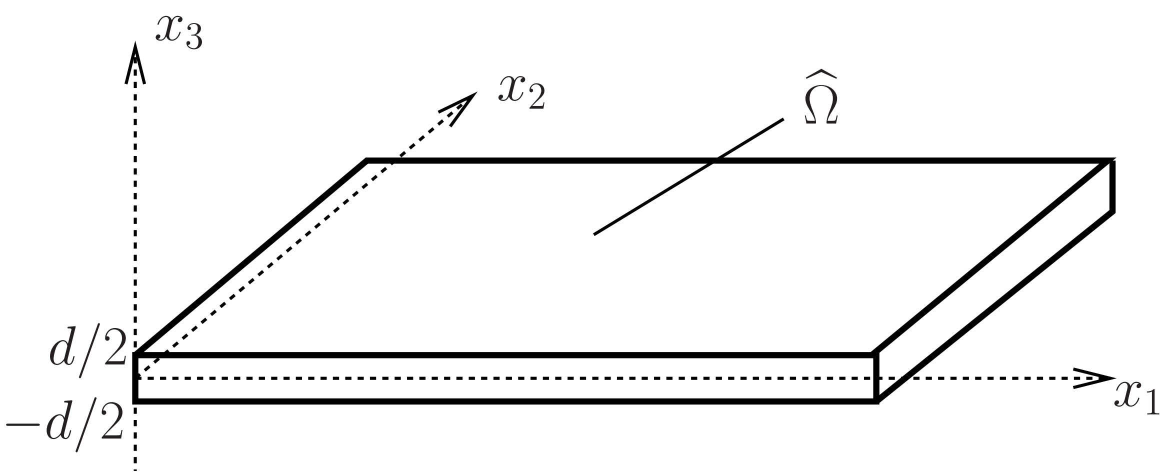
Elastic guided waves in plates

Geometry of a plate $\Omega := \widehat{\Omega} \times (-d/2, d/2)$



Elastic guided waves

are such harmonic solutions to the elastic wave equation

$$\rho\, \ddot{\mathbf{u}} - \operatorname{div} \boldsymbol{\sigma}(\mathbf{u}) = 0$$

that survive at large distances, e.g., the x_2-invariant waves propagating in x_1

$$\mathbf{u}(x,t) = \widehat{\mathbf{u}}(x_3)\, \mathrm{e}^{\mathrm{i}(\omega/c)(x_1 - ct)} = \widehat{\mathbf{u}}(x_3)\, \mathrm{e}^{\mathrm{i}(\xi x_1 - \omega t)}, \quad \text{where } \xi = \omega/c.$$

Elastic guided waves in plates

Shear horizontal (SH) waves

The only nonvanishing component is

$$u_2(x,t) = \widehat{u}_2(x_3)\, \mathrm{e}^{\mathrm{i}(\xi x_1 - \omega t)},$$

which leads to

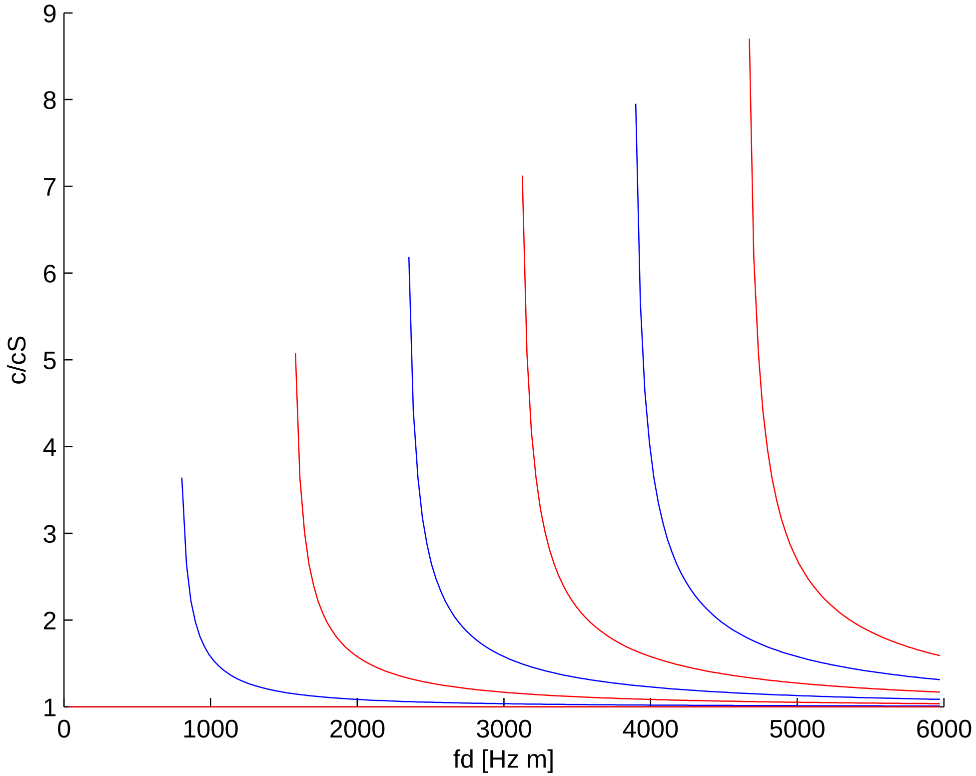
$$(c_S)^2\,(\widehat{u}_2'' - \xi^2\,\widehat{u}_2) = -\omega^2\,\widehat{u}_2 \quad \rightsquigarrow \quad \widehat{u}_2(x_3) = C\,\cos(\eta\, x_3) + D\,\sin(\eta\, x_3)$$

with $\eta^2 = \left(\frac{\omega}{c}\right)^2 = \left(\frac{\omega}{c_S}\right)^2 - \xi^2$, $(c_S)^2 = \frac{E}{2(1+\nu)\rho}$.

The constants C, D are determined as a nontrivial solution of the stress-free b.c. at $x_3 = \pm d$, which yields the symmetric $\widehat{u}_2(x_3) = \cos\left(\frac{\omega}{c} x_3\right)$ and the antisymmetric $\widehat{u}_2(x_3) = \sin\left(\frac{\omega}{c} x_3\right)$ SH modes with $\frac{\omega}{c} d = k\frac{\pi}{2}$, where the nonnegative integer k is odd in case of nonsymmetric modes.

Elastic guided waves in plates

Symmetric and antisymmetric SH dispersion curves



Elastic guided waves in plates

Theory of Lamb waves

Helmholtz decomposition of the displacement field

$\mathbf{u} = \nabla\Phi + \nabla\times\mathbf{H}$, where $\Phi(x_1, x_3; t) = f(x_3)\, \mathrm{e}^{\imath(\xi\, x_1 - \omega t)}$, $\mathbf{H}(x_1, x_3; t) = \mathbf{h}(x_3)\, \mathrm{e}^{\imath(\xi\, x_1 - \omega t)}$

leads to the pressure and shear wave equations

$$(c_\mathrm{P})^2\, \triangle\Phi = \ddot{\Phi}, \quad (c_\mathrm{S})^2\, \triangle\mathbf{H} = \ddot{\mathbf{H}} \text{ with } \nabla\cdot\mathbf{H} = 0,$$

respectively, where $(c_\mathrm{P})^2 = \frac{(1-\nu)E}{(1+\nu)(1-2\nu)\rho}$ and $(c_\mathrm{S})^2 = \frac{E}{2(1+\nu)\rho}$. Some of them read

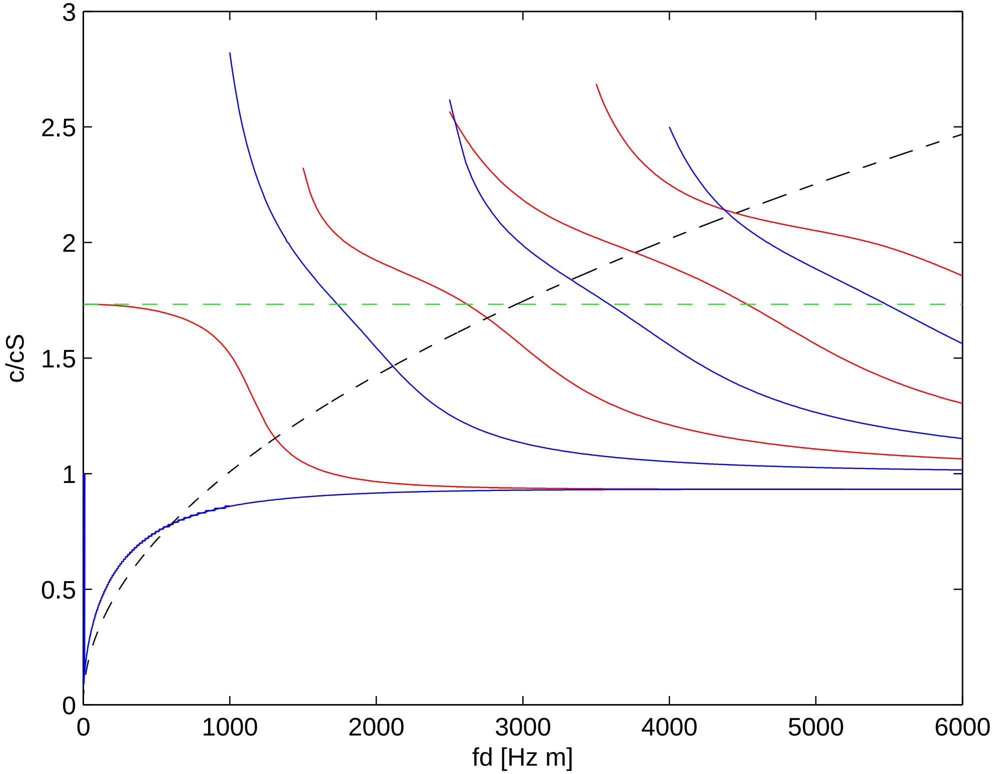
$$\begin{aligned} (c_\mathrm{P})^2\, (f'' - \xi^2\, f) = -\omega^2\, f \quad &\rightsquigarrow \quad f(x_3) = A\, \cos(\alpha\, x_3) + B\, \sin(\alpha\, x_3), \\ (c_\mathrm{S})^2\, (h_3'' - \xi^2\, h_3) = -\omega^2\, h_3 \quad &\rightsquigarrow \quad h_3(x_3) = G\, \cos(\beta\, x_3) + H\, \sin(\beta\, x_3) \end{aligned}$$

with $\alpha^2 = (\omega/c_\mathrm{P})^2 - \xi^2$, $\beta^2 = (\omega/c_\mathrm{S})^2 - \xi^2$.

The constants A, H are determined as a nontrivial solution of the stress-free b.c. at $z = \pm h/2$, which yields the characteristic equation for the symmetric Lamb waves. Similarly, determining B, G gives the antisymmetric Lamb waves.

Elastic guided waves in plates

Symmetric and antisymmetric Lamb dispersion curves



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Displacement finite elements for plates

Primal variational formulation of elastodynamics

$$\begin{cases} \rho \underbrace{\frac{\partial^2 u}{\partial t^2}(x,t)}_{=:\ddot{u}} - \mathrm{div}\sigma(x,t) = 0 & \text{for } x \in \Omega,\ t \in (0, t_{\max}), \\ \sigma(x,t) \cdot n(x) = T(x,t) & \text{for } x \in \Gamma,\ t \in \langle 0, t_{\max} \rangle, \\ u(x,0) = 0 & \text{for } x \in \overline{\Omega}, \\ \frac{\partial u}{\partial t}(x,0) = 0 & \text{for } x \in \overline{\Omega}, \end{cases}$$

reads to find $u \in L^2(0, t_{\max}; [H^1(\Omega)]^3)$ satisfying the operator differential equation

$$\rho\,\ddot{u} + L(u) = T, \quad u(0) = 0, \quad \dot{u}(0) = 0$$

in the space $L^2(0, t_{\max}; [H^{-1}(\Omega)]^3)$, where

$$\langle L(u), v \rangle := 2\,\mu \int_\Omega \varepsilon(u) : \varepsilon(v) + \lambda \int_\Omega \mathrm{div}(u)\,\mathrm{div}(v), \quad \langle T, v \rangle := \int_\Gamma T \cdot v.$$

Displacement finite elements for plates

Finite element semi-discretization

After a spatial finite element discretization we arrive at the system of ODEs

$$\mathbf{M}\,\ddot{\mathbf{u}} + \mathbf{K}\,\mathbf{u} = \mathbf{T},$$

where $\mathbf{M}$ and $\mathbf{K}$ denotes the mass and stifness matrix, respectively. The FE-solution reads

$$\mathbf{u}(t) = \sum_i \frac{1}{\sqrt{\lambda_i}} \int_0^t \sin\left(\sqrt{\lambda_i}(t-\tau)\right)(\mathbf{v}_i \otimes \mathbf{v}_i)\cdot\mathbf{T}(\tau)\,d\tau, \quad \text{with } \mathbf{K}\,\mathbf{v}_i = \lambda_i\,\mathbf{M}\,\mathbf{v}_i, \quad \|\mathbf{v}_i\| = 1.$$

Time discretization: explicit leap-frog, implicit Newmark

We shall rather employ the explicit time scheme: $0 = t_0 < t_1 < \dots,\ \Delta t = t_{k+1} - t_k$,

$$\mathbf{M}\,\ddot{\mathbf{u}}_k = -\mathbf{K}\,\mathbf{u}_{k-1} + \mathbf{T}_{k-1}, \quad \mathbf{u}_k := (\Delta t)^2\,\ddot{\mathbf{u}}_k + 2\,\mathbf{u}_{k-1} - \mathbf{u}_{k-2},$$

or the implicit, unconditionally stable Newmark scheme ($\beta := 1/4,\ \gamma := 1/2$):

$$\left\{\mathbf{M} + \beta(\Delta t)^2\mathbf{K}\right\}\ddot{\mathbf{u}}_{k+1} = \mathbf{F}_{k+1} - \mathbf{K}\,\mathbf{u}_{k+1/2}, \quad \mathbf{u}_{k+1} := \mathbf{u}_{k+1/2} + \beta\,(\Delta t)^2\,\ddot{\mathbf{u}}_{k+1}, \quad \dots$$

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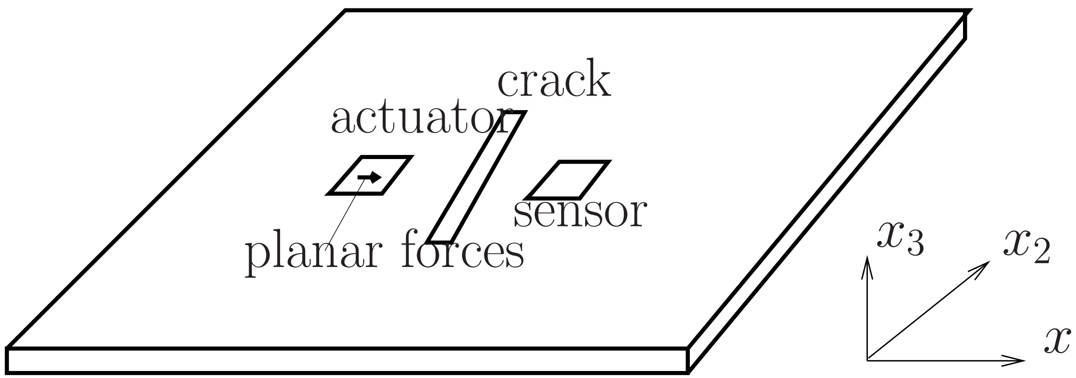
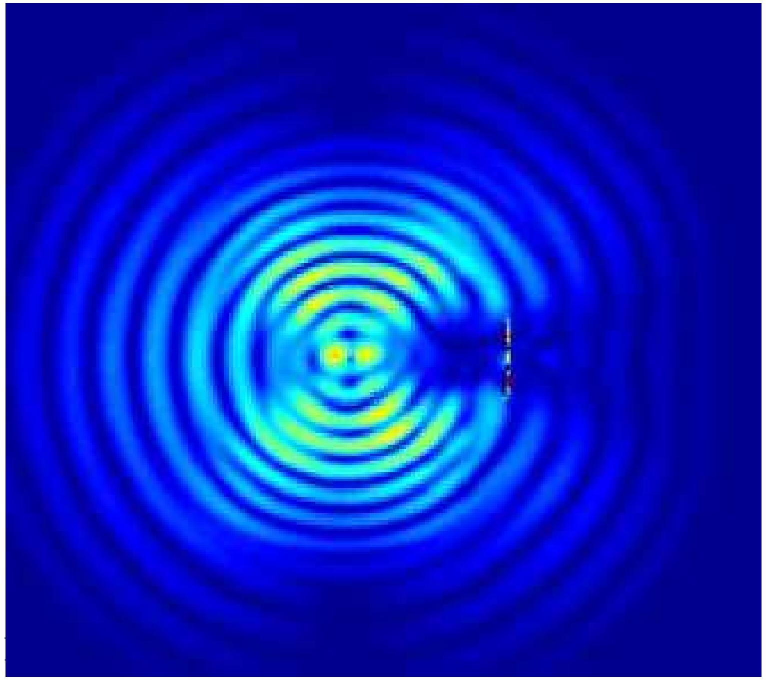
2d membranes and Kirchhoff's plates

Membranes, interaction of a wave with a crack

Planar stresses/forces ansatz: $\sigma_{ij}(x_1, x_2; t)$ for $i = 1, 2$; $\sigma_{i3} = \sigma_{3i} = 0$ leads to

$$u_i(x_1, x_2; t) \text{ for } i = 1, 2; \quad u_3(x_1, x_2, x_3; t) = x_3\, \varepsilon_{33}(x_1, x_2; t).$$





Time step solver: Cholesky or PCG iterations.

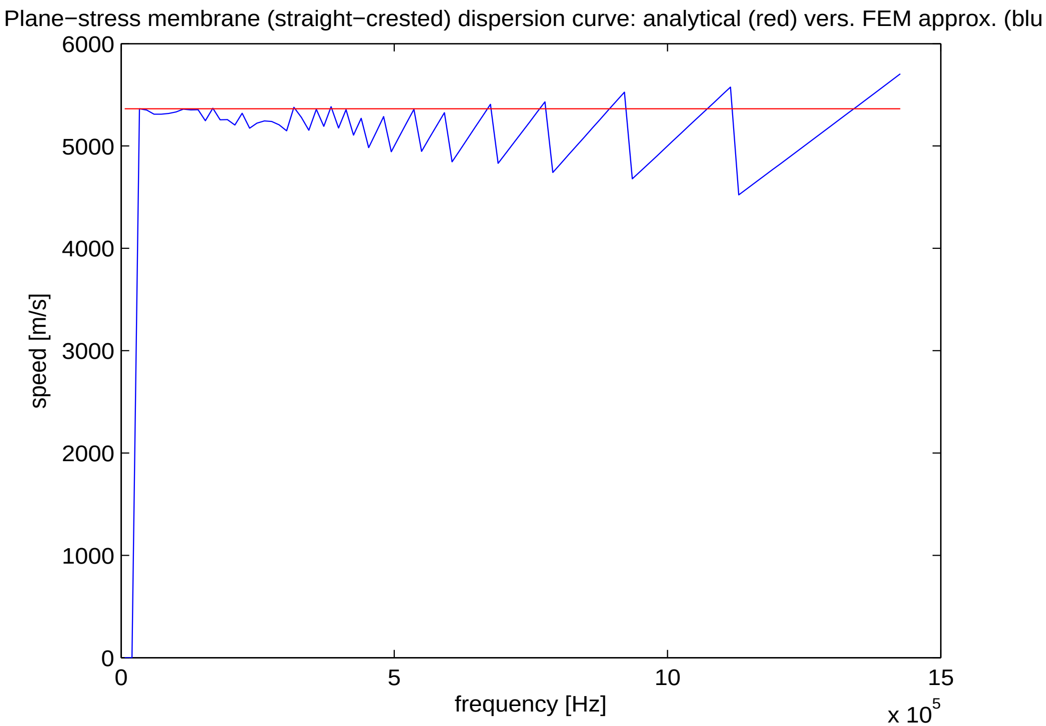
2d membranes and Kirchhoff's plates

Membranes: dispersion curve of a straight-crested wave

The straight-crested axial wave speed $c_{\mathrm{L}} = \sqrt{\frac{E}{(1-\nu^2)\rho}}$ is compared to a numerical counterpart relying on

$$\mathbf{P}^T \mathbf{K} \mathbf{P} \, \widetilde{\mathbf{v}}_i = \widetilde{\lambda}_i \, \mathbf{P}^T \mathbf{M} \, \mathbf{P} \, \widetilde{\mathbf{v}}_i,$$

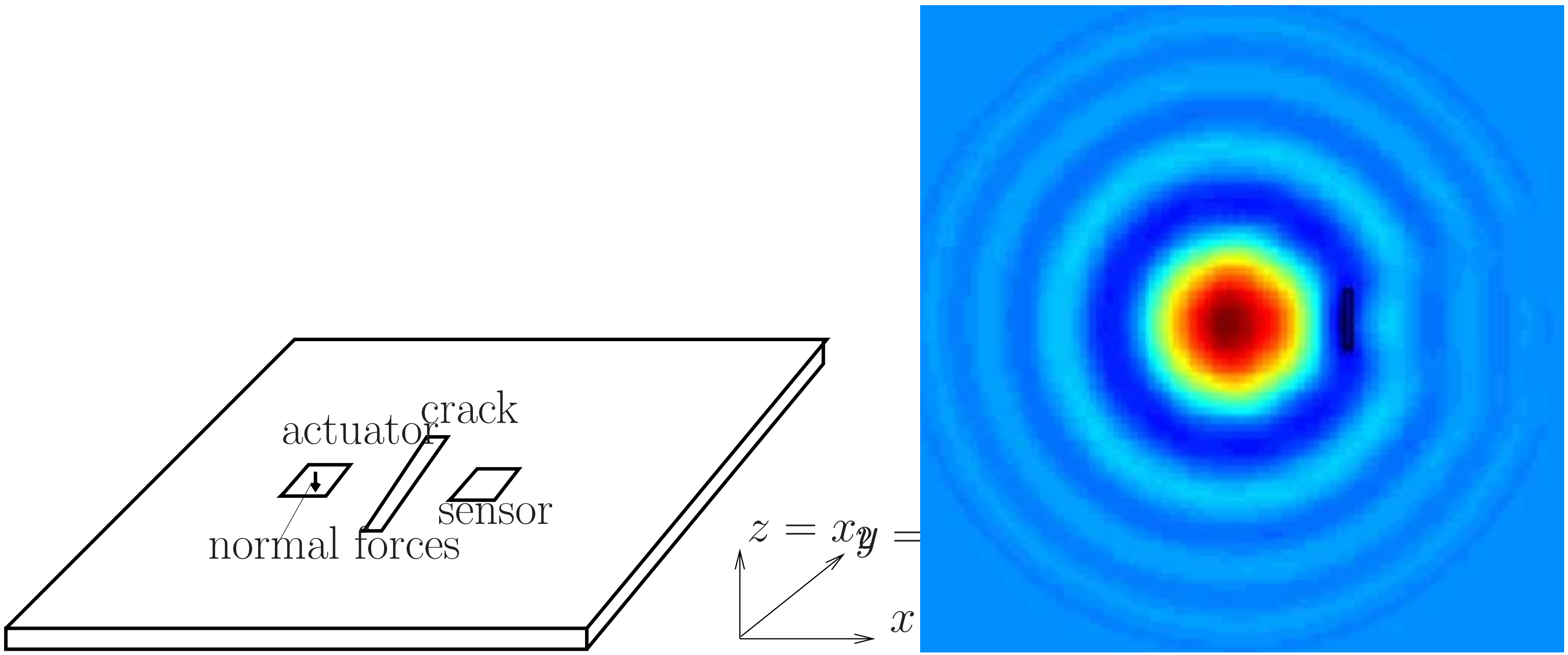
where $\mathbf{P}$ is the projection matrix (onto x_2-invariant functions).



2d membranes and Kirchhoff's plates

Kirchhoff's plates, interaction of a wave with a crack

Kirchhoff's ansatz: $u_i(x_1, x_2, x_3; t) = -x_3 \underbrace{\frac{\partial w}{\partial x_i}(x_1, x_2; t)}_{\text{rotations}}$ for $i = 1, 2$; $u_3(x_1, x_2; t) = w(x_1, x_2; t)$. DKT FEM [Batoz et al '80, LeTallec et al '95]



Time step solver: Cholesky or PCG iterations.

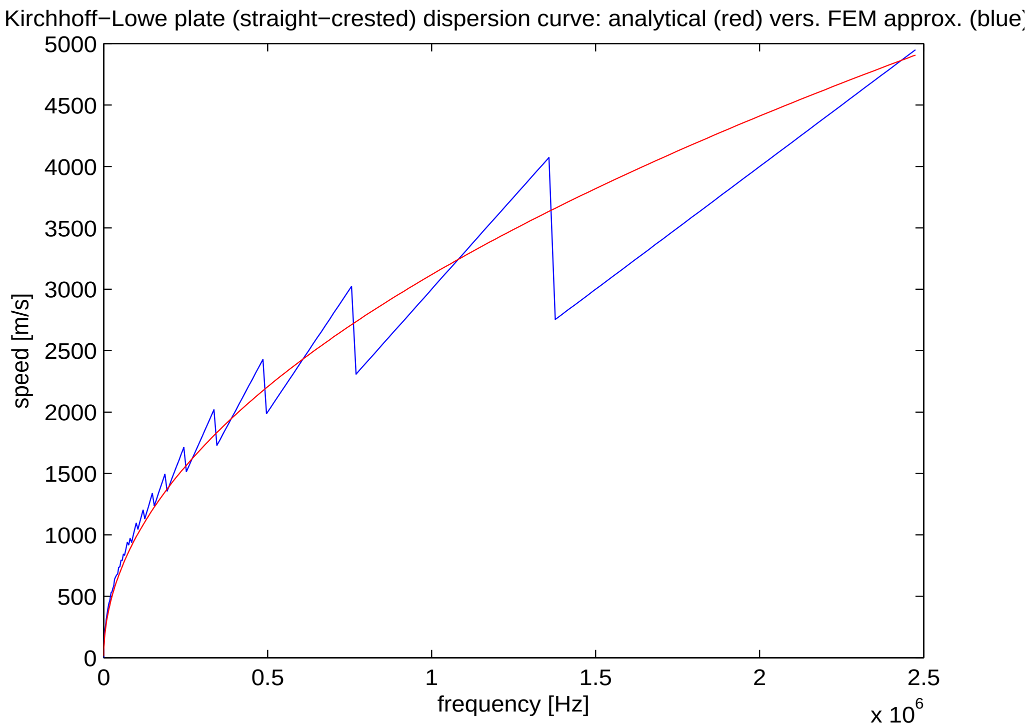
2d membranes and Kirchhoff's plates

Kirchhoff's plates: dispersion curve of a straight-crested wave

The straight-crested flexular wave speed $c_{\mathrm{F}}(df) = \sqrt[4]{\frac{E}{12\rho(1-\nu^2)}}\sqrt{2\pi df}$. is compared to numerics relying on

$$\mathbf{P}^T\mathbf{K}\mathbf{P}\,\widetilde{\mathbf{v}}_i = \widetilde{\lambda}_i\,\mathbf{P}^T\mathbf{M}\,\mathbf{P}\,\widetilde{\mathbf{v}}_i,$$

where $\mathbf{P}$ is the projection matrix (onto x_2-invariant functions).



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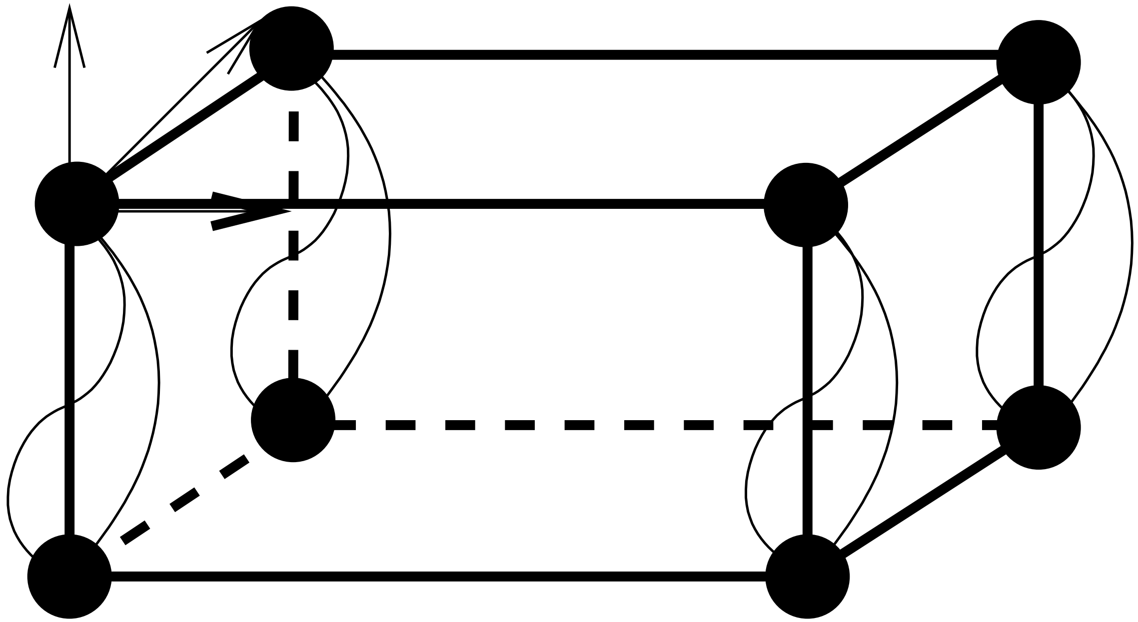
3d bricks enhanced with vertical edge polynomials

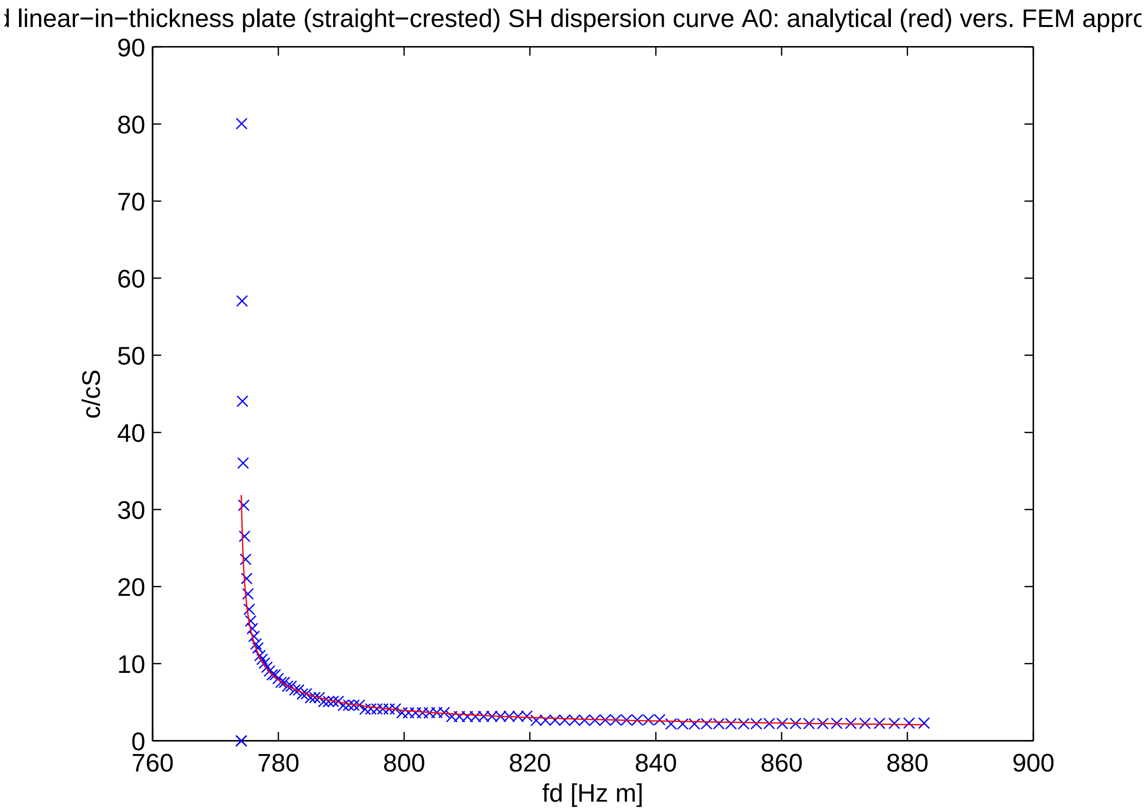
Example of bilinear-in-plane and cubic-in-thickness ($p = 3$) elements

$3 \cdot 8 = 24$ nodal $+ 3(p-1) \cdot 4 = 12(p-1)$ vertical edge DOFs



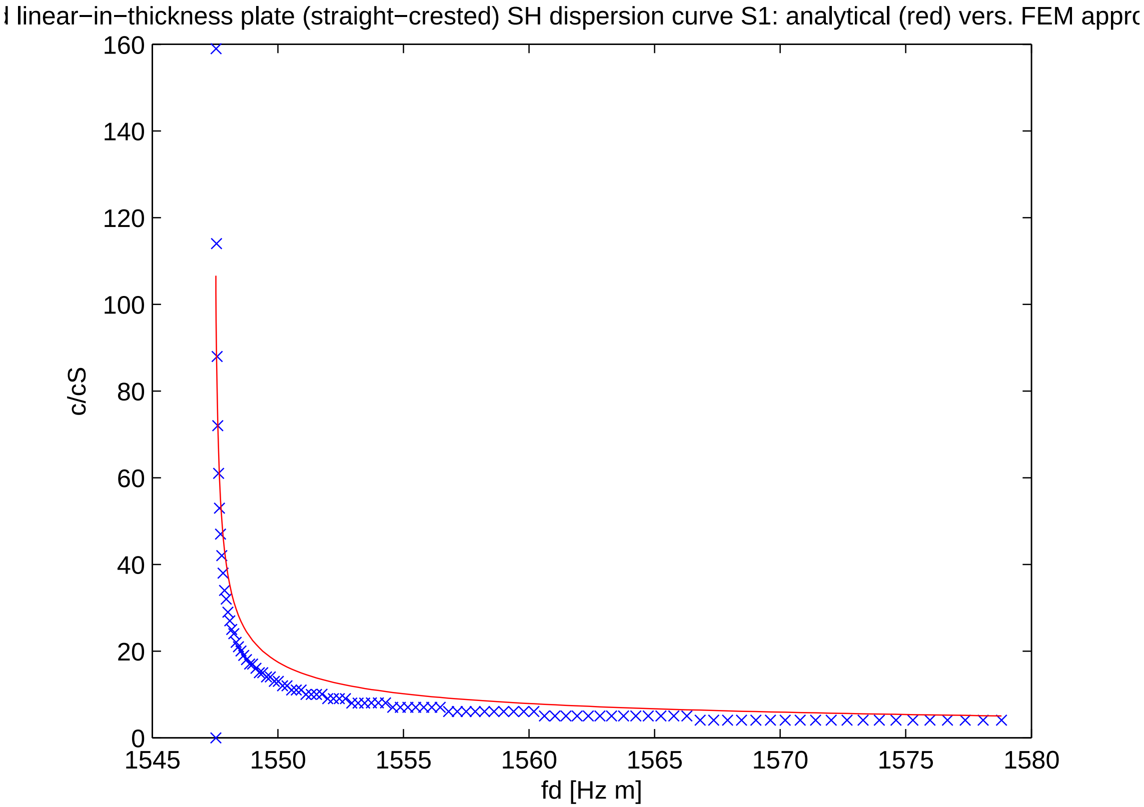
3d bricks enhanced with vertical edge polynomials

$p = 3$ includes SH A_0-mode



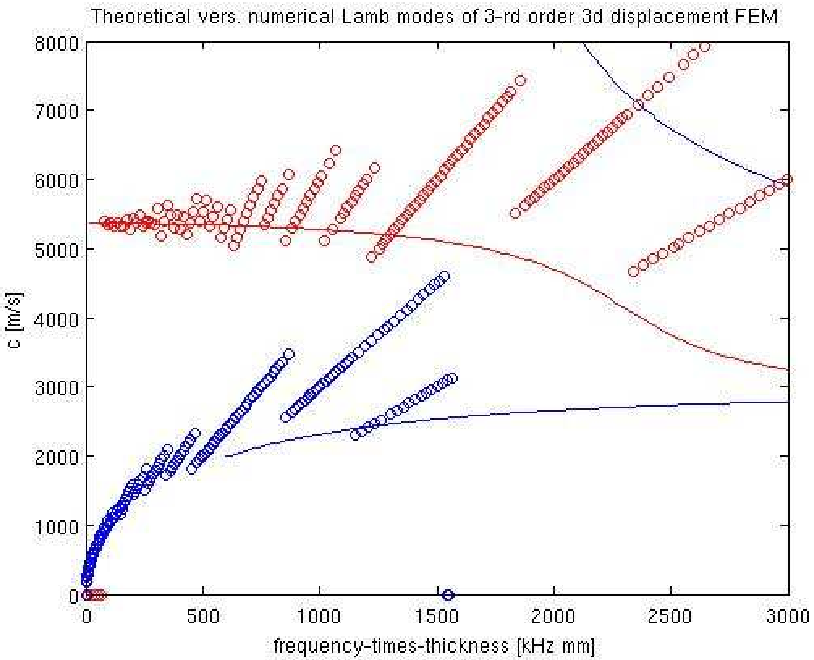
3d bricks enhanced with vertical edge polynomials

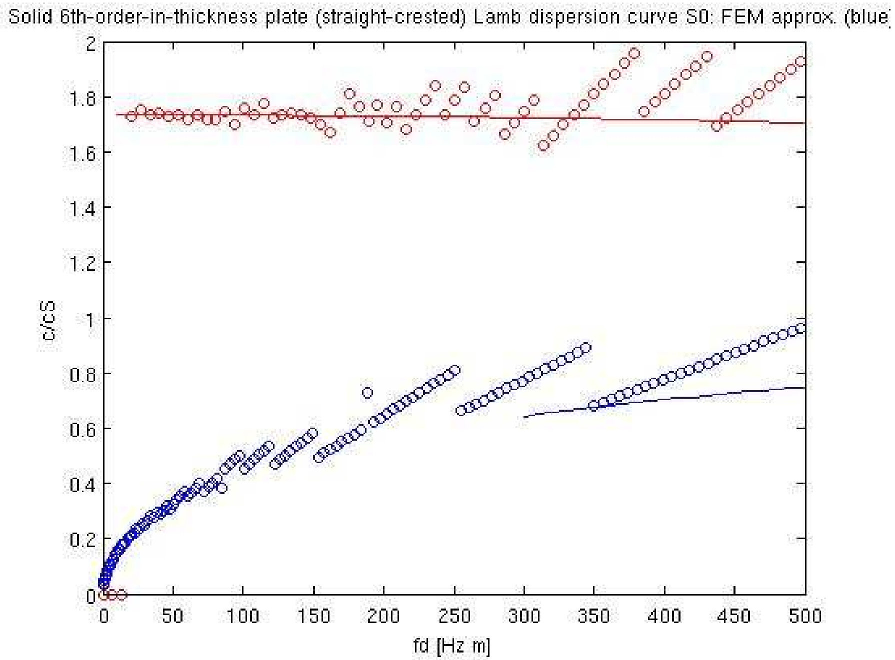
$p = 6$ includes SH S_1-mode



3d bricks enhanced with vertical edge polynomials

Neither $p = 3$ or $p = 6$ imitates Lamb modes

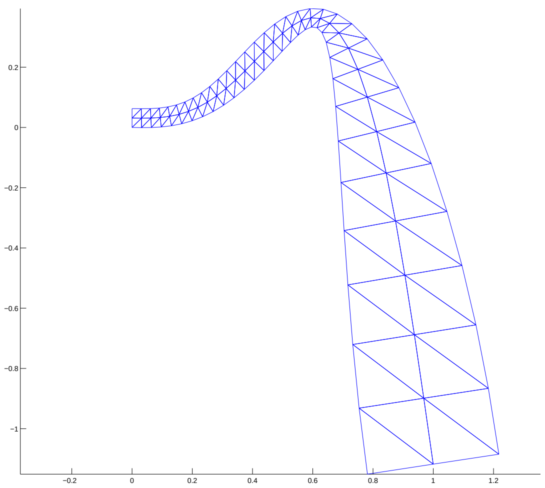


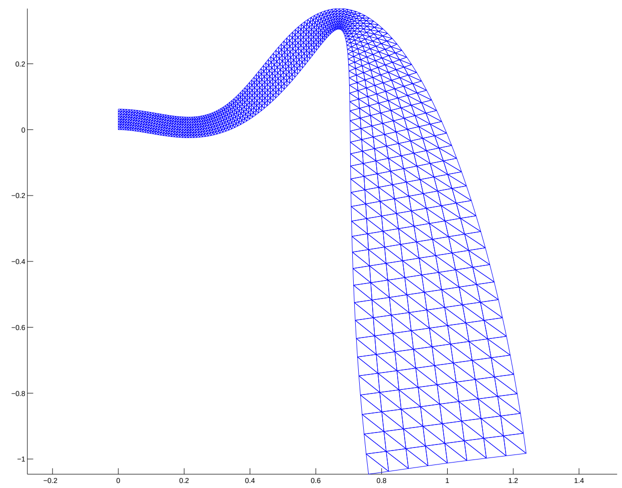


Displacement finite elements for plates

Shear locking effect (of bending modes?)

Convergence of 3d primal FEM suffers from $\frac{\operatorname{diam}\Omega}{d}$ entering Korn's ellipticity constant.





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Mixed elastic finite elements

Weak formulation of elasticity

Given (extensions of) boundary data (t, g) we shall find stresses $\sigma \in \Sigma_0 + t$ and displacements $u \in V_0 + g$ satisfying linearized Hooke's law

$$\langle A\,\sigma, \tau\rangle_{\Sigma^* \times \Sigma} - \langle \varepsilon(u), \tau\rangle_{\Sigma^* \times \Sigma} = 0 \quad \forall \tau \in \Sigma_0,$$

where $\varepsilon(u) := \frac{1}{2}(\nabla u + (\nabla u)^T)$, and the force equilibrium

$$\langle \operatorname{div} \sigma, v\rangle_{V^* \times V} = -\langle f, v\rangle_{V^* \times V} \quad \forall v \in V_0.$$

Primal formulation: $\Sigma := \Sigma_0 := [L^2(\Omega)]^{3\times 3}_{\mathrm{sym}}$, $V := [H^1(\Omega)]^3$

Since $\sigma = A^{-1}\,\varepsilon(u)$ pointwise, we are left to find $u \in V_g := \{u \in V \ : \ u = g \text{ on } \Gamma_{\mathrm{D}}\}$:

$$\int_\Omega A^{-1}\,\varepsilon(u) : \varepsilon(v) = \int_\Omega f \cdot v + \int_{\Gamma_{\mathrm{N}}} t \cdot v \quad \forall v \in V_0.$$

Mixed elastic finite elements

Mixed formulation: $\Sigma := [H(\mathrm{div};\Omega)]^{3\times 3}_{\mathrm{sym}}$, $V := V_0 := [L^2(\Omega)]^3$

Find $\sigma \in \Sigma_t := \{\sigma \in \Sigma \; : \; \sigma_n = t \text{ on } \Gamma_{\mathrm{N}}\}$ and $u \in V$:

$$\int_\Omega \sigma : \tau + \int_\Omega \operatorname{div} \tau \cdot u = \int_{\Gamma_{\mathrm{D}}} \tau_n \cdot g \qquad \forall \tau \in \Sigma_0,$$

$$\int_\Omega \operatorname{div} \sigma \cdot v = -\int_\Omega f \cdot v \qquad \forall v \in V$$

Schöberl & Pechstein (born Sinwel) ’09: $V := H(\mathrm{curl};\Omega) \rightsquigarrow \ldots$

Denote $H^{-1}(\Omega) := \left(H^1_{\Gamma_{\mathrm{D}},0}(\Omega)\right)^*$, $V_0 := \{v \in H(\mathrm{curl};\Omega) \; : \; v_t = 0 \text{ on } \Gamma_{\mathrm{D}}\}$. Then,

$$(V_0)^* = H^{-1}(\mathrm{div};\Omega) := \left\{q \in [H^{-1}(\Omega)]^3 \; : \; \operatorname{div} q \in H^{-1}(\Omega)\right\}.$$

The required regularity $\sigma \in [L^2(\Omega)]^{3\times 3}_{\mathrm{sym}}$, $\operatorname{div}\sigma \in H^{-1}(\mathrm{div};\Omega)$ leads to the new space

$$\ldots \rightsquigarrow \sigma \in \Sigma := H^{-1}(\operatorname{div}\operatorname{div};\Omega).$$

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Mixed TD-NNS finite elements [Schöberl & Pechstein]

$H^{-1}(\operatorname{div}\operatorname{div};\Omega)$-conformity $\approx$ continuity of normal-normal (NN) stresses

Distributional divergence. For $v \in [C_0^\infty(\Omega)]^3$:

$$\langle \operatorname{div}\sigma, v\rangle_{V^*\times V} := -\int_\Omega \sigma : \nabla v = \sum_{\text{elems } T} \left\{ \int_T \operatorname{div}\sigma \cdot v - \int_{\partial T} \sigma_{nt}\cdot v_t \right\} - \sum_{\text{faces } F} \int_F \underbrace{[\sigma_{nn}]}_{=0} v_n$$

$$\leq \sum_{\text{elems } T} \left\{ \|\operatorname{div}\sigma\|_{0,T}\, \|v\|_{0,T} + \|\sigma_{nt}\|_{\frac{1}{2},\partial T}\, \|v_t\|_{-\frac{1}{2},\partial T} \right\} \leq C(\sigma)\, \|v\|_{\text{curl},\Omega}.$$

By density, the continous functional can be extended to $H(\text{curl};\Omega)$.

Mixed TD-NNS formulation: $\Sigma := H^{-1}(\operatorname{div}\operatorname{div};\Omega)$, $V := H(\text{curl};\Omega)$

Find $\sigma \in \Sigma_{t_n} := \{\sigma \in \Sigma : \sigma_{nn} = t_n \text{ on } \Gamma_{\text{N}}\}$, $u \in V_{g_t} := \{u \in V : u_t = g_t \text{ on } \Gamma_{\text{D}}\}$:

$$\int_\Omega \sigma : \tau + \langle \operatorname{div}\tau, u\rangle_{V^*\times V} = \int_{\Gamma_{\text{D}}} \tau_{nn}\, g_n \qquad \forall \tau \in \Sigma_0,$$

$$\langle \operatorname{div}\sigma, v\rangle_{V^*\times V} = -\int_\Omega f\cdot v + \int_{\Gamma_{\text{N}}} t_t \cdot v_t \qquad \forall v \in V_0$$

Mixed TD-NNS finite elements [Schöberl & Pechstein]

Abstract theory of mixed formulations, inf-sup condition

H-spaces Σ, V, $B \in \mathcal{L}(\Sigma, V^*)$, $A \in \mathcal{L}(\Sigma, \Sigma^*)$ is $\operatorname{Ker} B$-elliptic, $g \in \Sigma^*$, $f \in \operatorname{Im} B$.

$$\begin{aligned} A\,\sigma + B^T\,u &= g \text{ on } \Sigma^* && (1)\\ B\,\sigma &= f \text{ on } V^* && (2) \end{aligned}$$

From (2): $\sigma = \sigma_o + \sigma_f$, where $B\,\sigma_f = f$, and from (1): $\sigma_0 \in \operatorname{Ker} B$ uniquely solves

$$\langle A\,\sigma_0, \tau_0\rangle_{\Sigma^*\times\Sigma} = \langle g - A\,\sigma_f, \tau_0\rangle_{\Sigma^*\times\Sigma} \quad \forall \tau_0 \in \operatorname{Ker} B.$$

It remains to find $u \in V$:

$$\langle B\,\tau_\perp, u\rangle_{V^*\times V} = -\langle A\,\sigma, \tau_\perp\rangle_{\Sigma^*\times\Sigma} \quad \forall \tau_\perp \in (\operatorname{Ker} B)^\perp.$$

The existence follows from the $(\operatorname{Ker} B)^\perp$-coercivity, the so-called inf-sup condition

$$\|B\,\sigma\|_{\Sigma^*} \geq \delta\,\|v\|_V.$$

Mixed TD-NNS finite elements [Schöberl & Pechstein]

Displacements: Nédélec-II hexahedron

Consider hexahedron $T := (0, h_x) \times (0, h_y) \times (0, h_z)$ with (potentially) $h_x, h_y \gg h_z$.
We choose the lowest-order Nédelec-II $H(\mathrm{curl}; \Omega)$-conforming element:

$$v_x, v_y \in P^1 \otimes P^1 \otimes P^2 \text{ and } v_z \in P^2 \otimes P^2 \otimes P^1.$$

Tangential continuity $\rightsquigarrow$ 2 DOFs per edge, 4 DOFs per vertical face, and 2 bubbles.

Stresses: discrete inf-sup cond. $\rightsquigarrow$ stable TD-NNS hexahedron [L.'16]

Analysis of the discrete inf-sup condition in the discrete broken norms

$$\|u\|^2_{V^h} := \sum_{\text{elems } T} \|\varepsilon(u)\|^2_{0,T} + \sum_{\text{faces } F} \frac{1}{h_F} \|[u_n]\|^2_{0,F}, \quad \|\sigma\|^2_{\Sigma^h} := \sum_{\text{elems } T} \|\sigma\|^2_{0,T} + \sum_{\text{faces } F} \|\sigma_{nn}\|_{0,F}$$

and normal-normal continuity yield 6 DOFs per vertical face, 9 DOFs per horizontal face, and 57 bubbles.

The analysis of S.&P. gets rid of Korn's inequality and h_x/h_z, i.e., locking-free elements.

Mixed TD-NNS finite elements [Schöberl & Pechstein]

Mixed TD-NNS variational formulation of elastodynamics

reads to find $\sigma \in L^2(0, t_{\max}; \underbrace{H(\operatorname{div}\operatorname{div};\Omega)}_{=:\Sigma})$ and $u \in L^2(0, t_{\max}; \underbrace{H(\operatorname{curl};\Omega)}_{=:V})$ satisfying

$$u(0) = 0, \quad \dot{u}(0) = 0, \quad \sigma_{nn} = T_n \text{ on } \Gamma$$

and

$$\int_\Omega A\sigma : \tau + \langle \operatorname{div} \tau, u \rangle_V = 0; \quad \langle \operatorname{div} \sigma, v \rangle - \rho \, \langle \ddot{u}, v \rangle = \int_\Gamma T_t \cdot v_t$$

for all $\tau \in L^2(0, t_{\max}; \Sigma_0)$ and $v \in L^2(0, t_{\max}; V)$.

Finite element semi-discretization and the Newmark method

lead to find $\boldsymbol{\sigma}_{k+1} := \boldsymbol{\sigma}^H_{k+1} + \boldsymbol{\sigma}^P(\mathbf{T}_{k+1})$ and $\mathbf{u}_{k+1}$ such that

$$\begin{pmatrix} \frac{1}{\beta\,(\Delta t)^2}\mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & -\mathbf{M} \end{pmatrix} \begin{pmatrix} \boldsymbol{\sigma}^H_{k+1} \\ \ddot{\mathbf{u}}_{k+1} \end{pmatrix} = \begin{pmatrix} \frac{1}{\beta\,(\Delta t)^2}\left(-\mathbf{A}\,\boldsymbol{\sigma}^P(\mathbf{T}_{k+1}) - \mathbf{B}\,\ddot{\mathbf{u}}_{k+1/2}\right) \\ -\mathbf{B}^T\,\boldsymbol{\sigma}^P(\mathbf{T}_{k+1}) \end{pmatrix},$$

$$\mathbf{u}_{k+1} := \mathbf{u}_{k+1/2} + \beta\,(\Delta t)^2\,\ddot{\mathbf{u}}_{k+1}, \quad \ldots$$

Elastodynamics of Cracked Plates
Using Mixed Finite Elements

Outline

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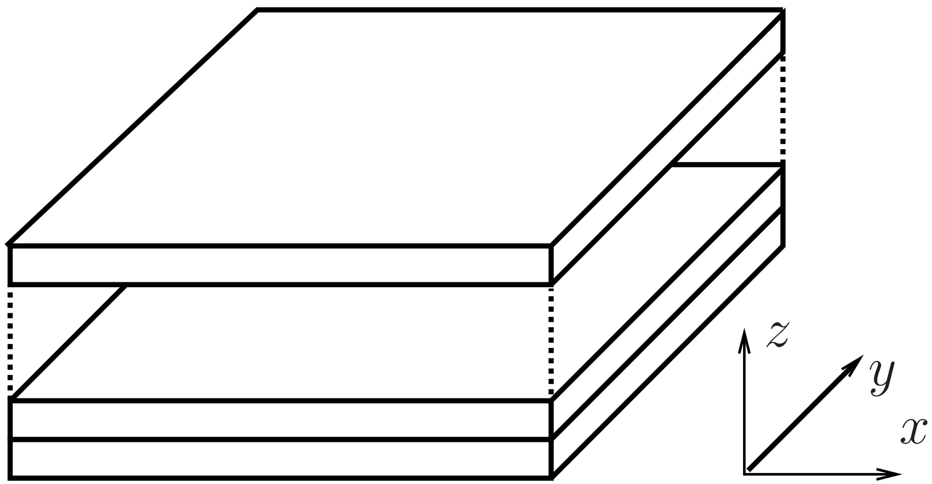
Numerical dispersion analysis [L. & Schöberl '16]

Eigenvalue quasi-periodic problem on 1 element $\rightsquigarrow$ dispersion $\kappa(\omega)$

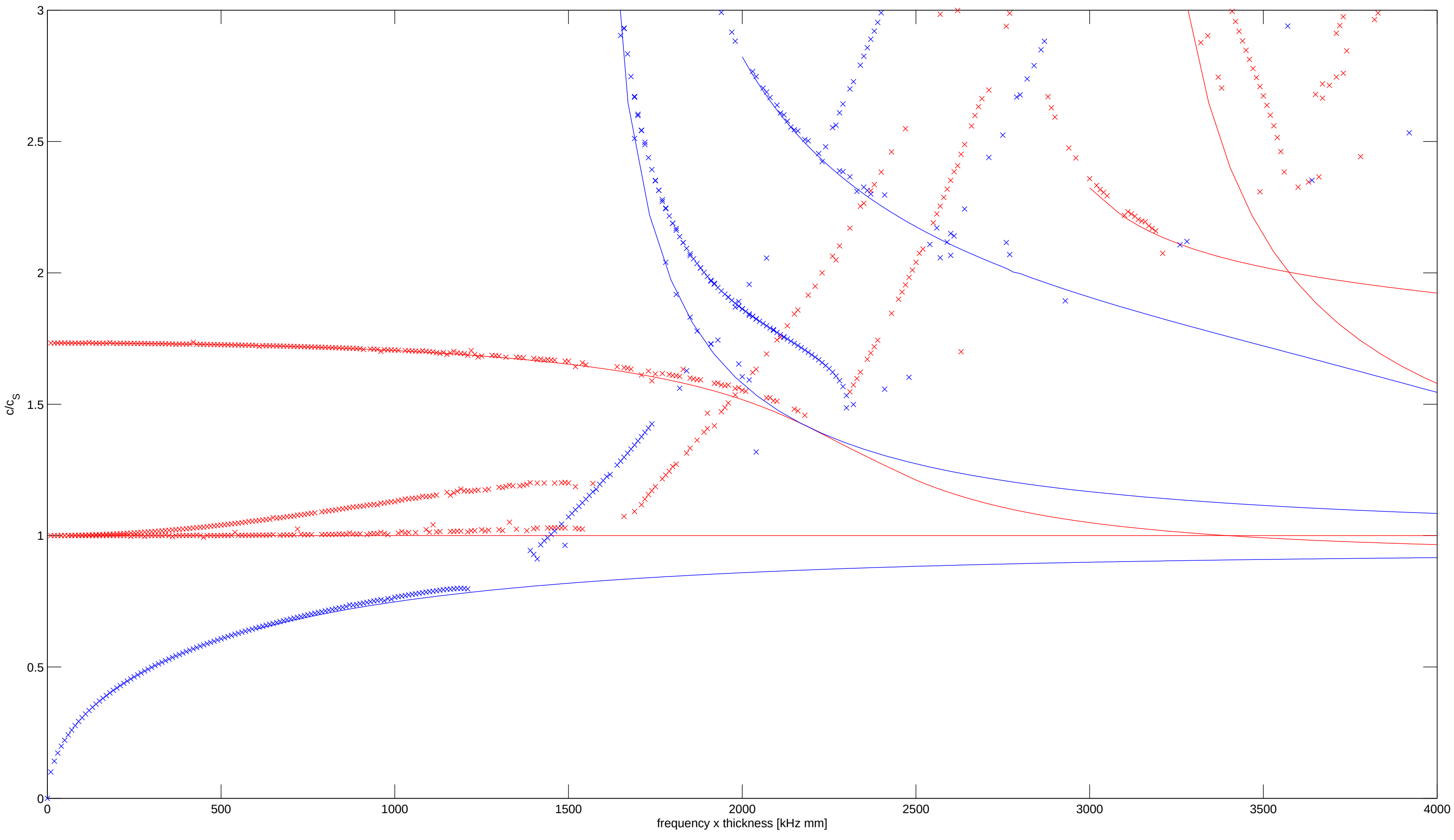
Find $\lambda = \mathrm{e}^{\imath\,\kappa\,h_x} \in \mathbb{C}$ and (σ, u):



$$\begin{aligned}
\text{mixed elasticity :}& \\
A\,\sigma + B^T\,u &= 0,\\
B\,\sigma + \omega^2\,M\,u &= 0,\\
+\text{ quasi-periodic b.c. in } x :& \\
u(0,y,z) &= \lambda\,u(h,y,z),\\
\sigma_n(0,y,z) &= -\lambda\,\sigma_n(h,y,z),\\
+\text{ periodic b.c. in } y :& \\
u(x,0,z) &= u(x.h,z),\\
\sigma_n(x,0,z) &= -\sigma_n(x,h,z),\\
+\text{ sym./antisym. b.c. in } z :& \\
u(x,y,0) &= \pm u(x,y,h_z),\\
\sigma_n(x,y,0) &= \sigma_n(x,y,h_x) = 0.
\end{aligned}$$

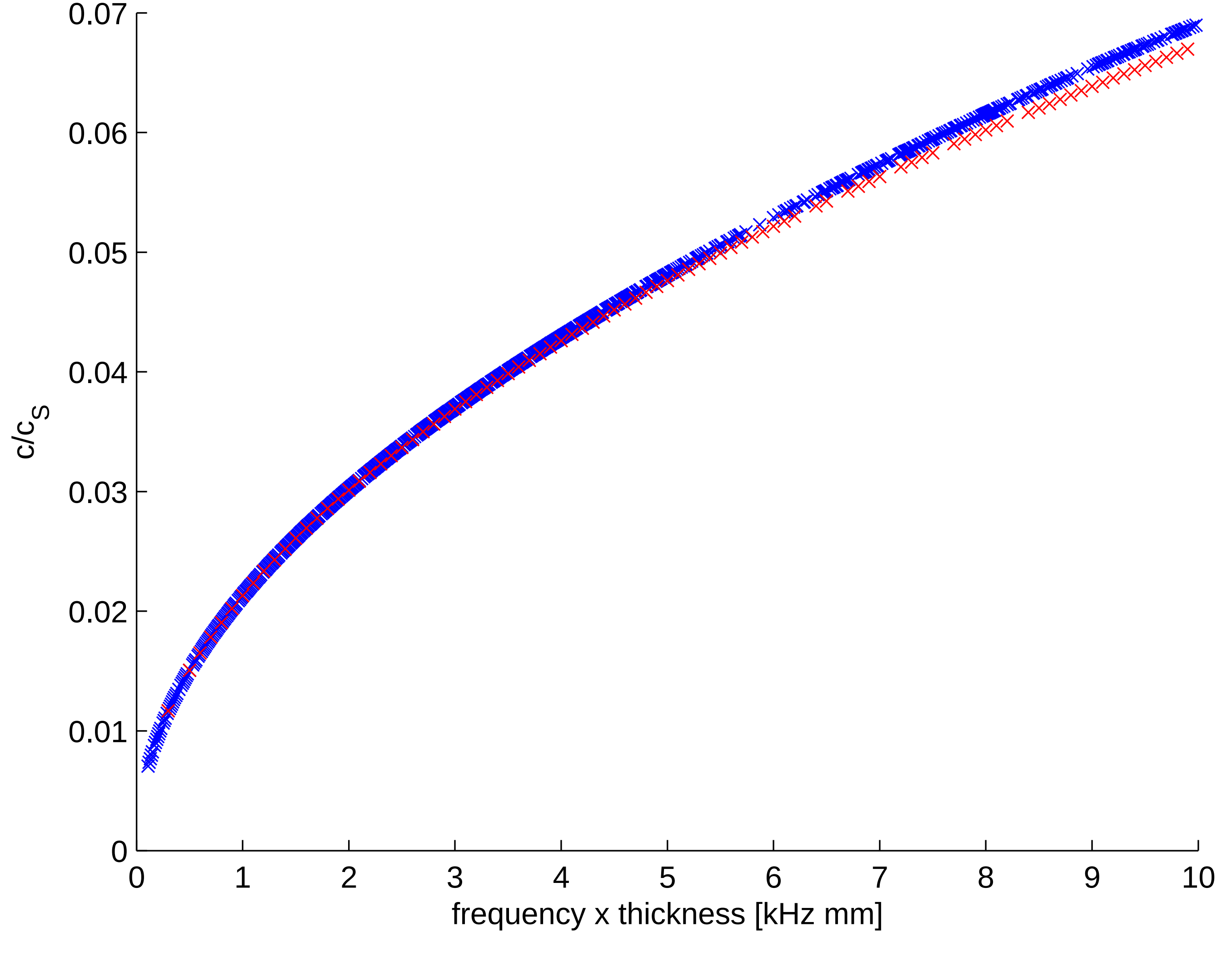
Numerical dispersion analysis [L. & Schöberl '16]

Numerical (3 layers) vers. analytical (solid lines) dispersion analysis



Numerical dispersion analysis [L. & Schöberl '16]

Robustness w.r.t. thickness: 1x1x0.1 mm (blue) vers. 1x1x1 mm (red)



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Mixed TD-NNS tensor-product elements

3d tensor-product TD-NNS: 24+16+2 displs., 0+42+57 stresses

Let's get rid of stress (57 bubbles).

Mixed TD-NNS tensor-product elements

Hybridization

$$\begin{pmatrix} \frac{1}{\beta\,(\Delta t)^2}\,\mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & -\mathbf{M} \end{pmatrix} \begin{pmatrix} \boldsymbol{\sigma}^H_{k+1} \\ \ddot{\mathbf{u}}_{k+1} \end{pmatrix} = \begin{pmatrix} \frac{1}{\beta\,(\Delta t)^2}\,\left(-\mathbf{A}\,\boldsymbol{\sigma}^P(\mathbf{T}_{k+1}) - \mathbf{B}\,\ddot{\mathbf{u}}_{k+1/2}\right) \\ -\mathbf{B}^T\,\boldsymbol{\sigma}^P(\mathbf{T}_{k+1}) \end{pmatrix},$$

We shall tear and interconnect the stress DOFs so that

- the stresses are left elementwise discontinuous,
- the continuity across faces is re-inforced by additional Lagrange multipliers $\boldsymbol{\lambda}$ that are related to normal displacements,
- and the stresses are statically condensated.

This leads to a new purely displacement spd system (elementwise assembly):

$$\begin{pmatrix} \mathbf{M} + \beta\,(\Delta t)^2\,\widetilde{\mathbf{B}}^T\,\widetilde{\mathbf{A}}^{-1}\,\widetilde{\mathbf{B}} & , \ \beta\,(\Delta t)^2\,\widetilde{\mathbf{B}}^T\,\widetilde{\mathbf{A}}^{-1}\,\mathbf{C} \\ \beta\,(\Delta t)^2\,\mathbf{C}^T\,\widetilde{\mathbf{A}}^{-1}\,\widetilde{\mathbf{B}} & , \ \beta\,(\Delta t)^2\,\mathbf{C}^T\,\widetilde{\mathbf{A}}^{-1}\,\mathbf{C} \end{pmatrix} \begin{pmatrix} \ddot{\mathbf{u}}_{k+1} \\ \boldsymbol{\lambda}_{k+1} \end{pmatrix} = \begin{pmatrix} \cdot \\ \cdot \end{pmatrix}$$

with the sign (interconnecting face DOFs) matrix $\mathbf{C}$.

Mixed TD-NNS tensor-product elements

Parallel preconditioning

Since $\Delta t \approx 10^{-7}$, the matrix

$$\begin{pmatrix} \mathbf{M} + \beta\,(\Delta t)^2\,\widetilde{\mathbf{B}}^T\,\widetilde{\mathbf{A}}^{-1}\,\widetilde{\mathbf{B}} & , & \beta\,(\Delta t)^2\,\widetilde{\mathbf{B}}^T\,\widetilde{\mathbf{A}}^{-1}\,\mathbf{C} \\ \beta\,(\Delta t)^2\,\mathbf{C}^T\,\widetilde{\mathbf{A}}^{-1}\,\widetilde{\mathbf{B}} & , & \beta\,(\Delta t)^2\,\mathbf{C}^T\,\widetilde{\mathbf{A}}^{-1}\,\mathbf{C} \end{pmatrix}$$

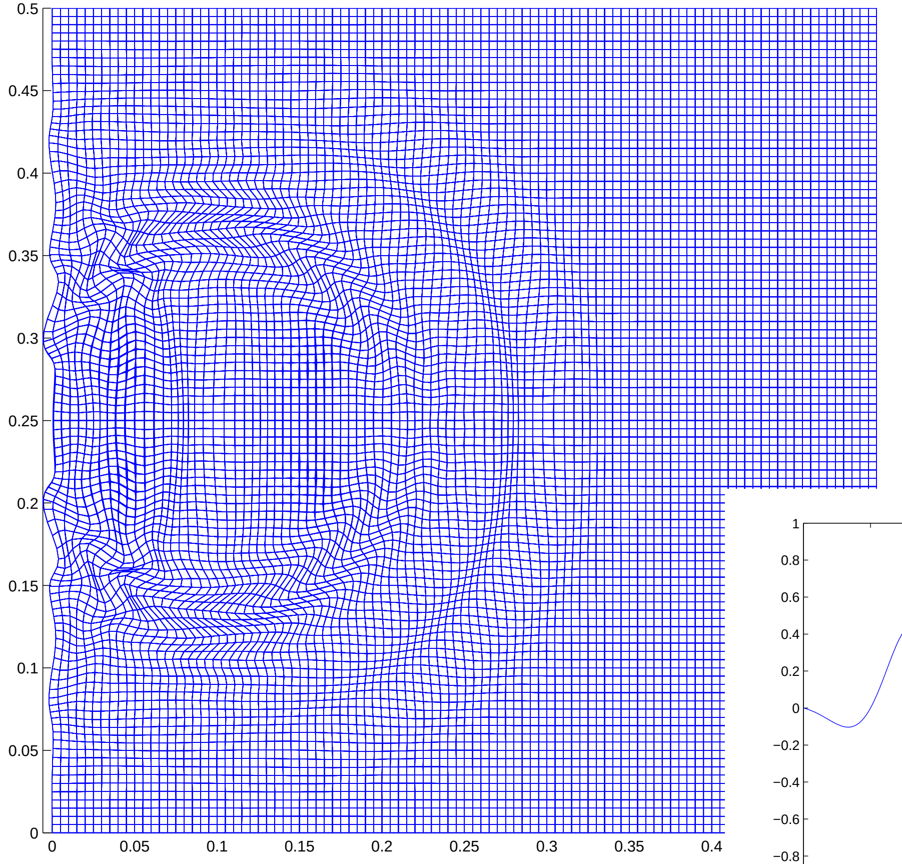
is mass dominant. When the Matlab Cholesky decomposition fails, we employ PCG preconditioned as follows:

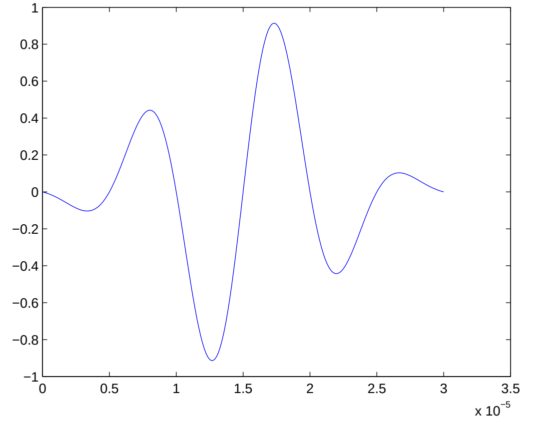
- 3 multiplicative (Richardson) smoothing steps.
- One smoothing consists of N (number of cores) additive (overlapping Schwarz) nested smoothers.
- The nested smoother comprises, as proposed by Joachim Schöberl, solutions to local patch subproblems around vertical edges, assembled multiplicatively(Gauss-Seidel).

Early experiments in 3d, 1M DOFs, rel. prec. 10^{-9}:
$N = 1 \rightsquigarrow 5$ PCG iterations,
$N = 24 \rightsquigarrow 10$ PCG iterations.

Mixed TD-NNS tensor-product elements

Interaction of short-pulse waves with surface cracks





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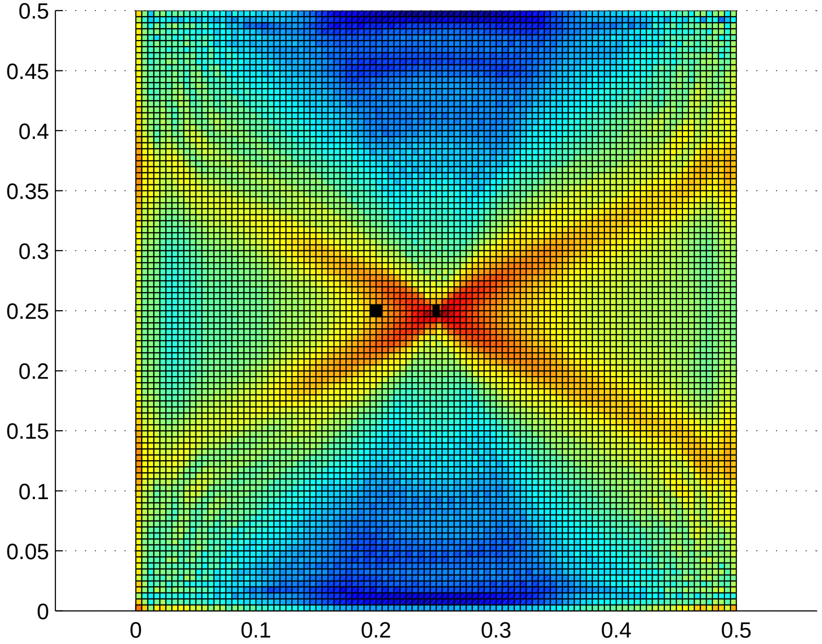
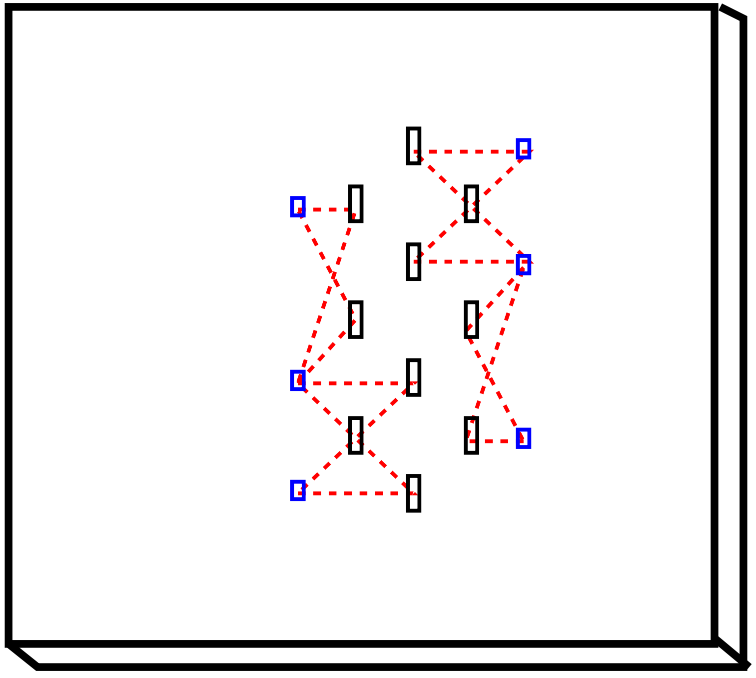
Mixed TD-NNS tensor-product elements

Damage sensitivity, optimization of actuator-crack-sensor trajectories





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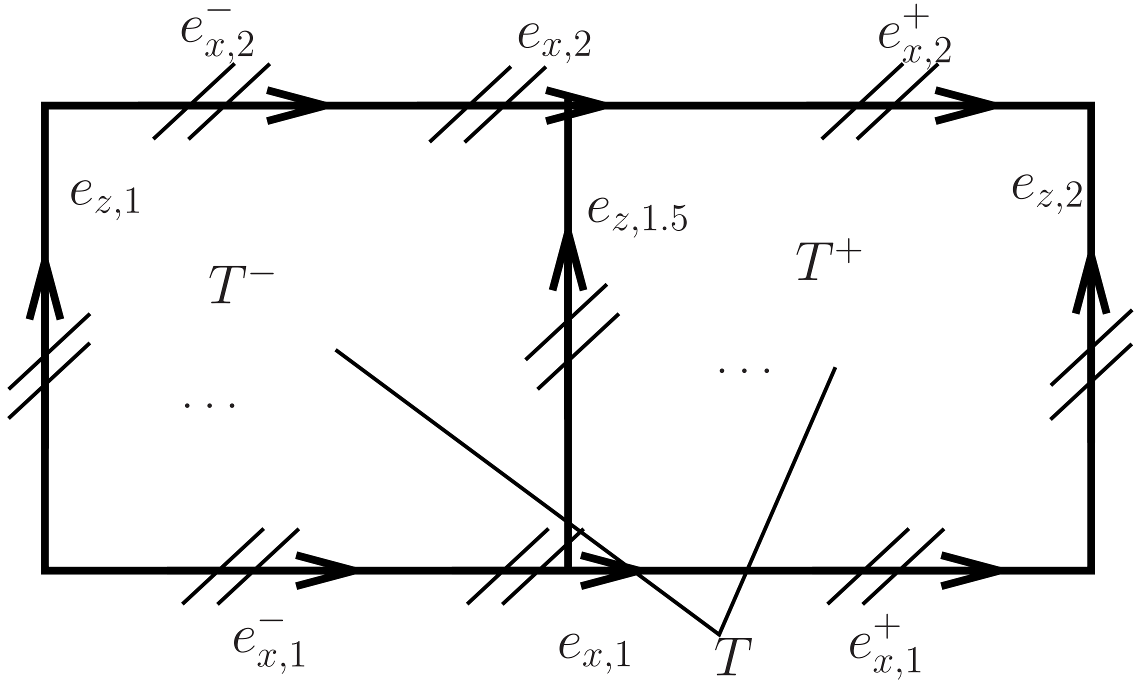
Elastodynamics of Cracked Plates Using Mixed Finite Elements

Outlook: Multigrid for hybridized tensor-product TD-NNS

Interpolation $\mathbf{I}_{H,h}$ via the natural embedding, $H = 2h$, $\boldsymbol{\mathcal{A}}^h := (\mathbf{I}_{H,h})^T \boldsymbol{\mathcal{A}}^H \mathbf{I}_{H,h}$.

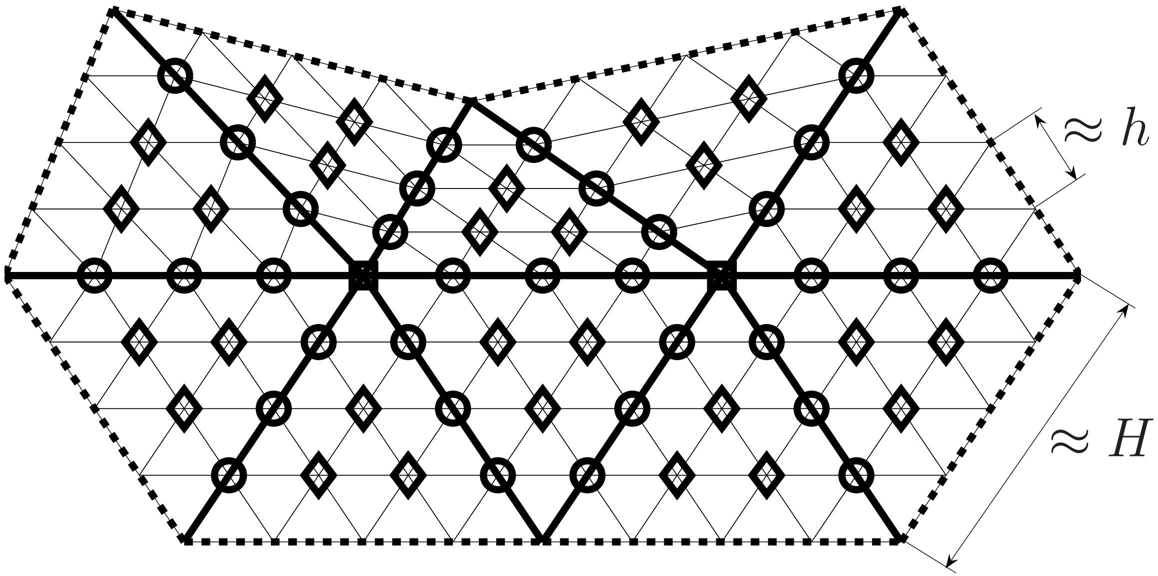


$$\varphi^{x,H}_{e_x,m,0} = \frac{1}{2}\left(\varphi^{x,h}_{e_x^-,m,0} + \varphi^{x,h}_{e_x^+,m,0}\right), \quad \varphi^{x,H}_{e_x,m,1} = -\frac{1}{4}\varphi^{x,h}_{e_x^-,m,0} + \frac{1}{4}\varphi^{x,h}_{e_x^-,m,1} + \frac{1}{4}\varphi^{x,h}_{e_x^+,m,0} + \frac{1}{4}\varphi^{x,h}_{e_x^+,m,1},$$

$$\varphi^{z,H}_{e_z,m,i} = \varphi^{z,h}_{e_z,m,i} + \frac{1}{2}\varphi^{z,h}_{e_z,1.5,i}, \quad \varphi^{x,H}_{T,0} = \frac{1}{2}\left(\varphi^{x,h}_{T^-,0} + \varphi^{x,h}_{T^+,0}\right), \quad \dots$$

Elastodynamics of Cracked Plates Using Mixed Finite Elements

Outlook: Primal DDM [BPS'86; LBVM'15] for tensor-prod. TD-NNS



$$\overline{\Omega} = \bigcup_{i=1}^{N} \overline{\Omega_i}, \quad \Gamma := \bigcup_{i=1}^{N} \partial\Omega_i \setminus \partial\Omega,$$

$\diamond$ . . . subdomain interior nodes (I) $\rightsquigarrow V_{\mathrm{I}}$

$\circ$ . . . edge interior nodes (E) $\Big\}\rightsquigarrow V_\Gamma$

$\square$ . . . vertex coarse nodes (V)

$$V = (\underbrace{V_1 \oplus_a \cdots \oplus_a V_N}_{=:V_{\mathrm{I}}}) + (\underbrace{V_{\mathrm{E}} + V_{\mathrm{V}}}_{=:V_\Gamma}), \quad \mathbf{A} = \begin{pmatrix} \mathbf{A}_{\mathrm{II}} & \mathbf{A}_{\mathrm{I}\Gamma} \\ \mathbf{A}_{\Gamma\mathrm{I}} & \mathbf{A}_{\Gamma\Gamma} \end{pmatrix}, \quad \mathbf{S} := \mathbf{A}_{\Gamma\Gamma} - \mathbf{A}_{\Gamma I}\mathbf{A}_{II}^{-1}\mathbf{A}_{I\Gamma}$$

$$\mathbf{S} = \begin{pmatrix} \mathbf{S}_{\mathrm{EE}} & \mathbf{S}_{\mathrm{EV}} \\ \mathbf{S}_{\mathrm{VE}} & \mathbf{S}_{\mathrm{VV}} \end{pmatrix} \approx \widehat{\mathbf{S}} := \mathbf{I} \begin{pmatrix} \mathrm{blkdiag}(\mathbf{S}_{\mathrm{EE}}) & \mathbf{0} \\ \mathbf{0} & \mathbf{A}^H \end{pmatrix} \mathbf{I}^T, \quad \mathrm{cond}(\widehat{\mathbf{S}}^{-1}\mathbf{S}) = O\left((1 + \log(H/h))^2\right).$$